Head Office: 2nd Floor, Grand Plaza, Fraser Road, Dak Bunglow, Patna - 01

JEE Main 2023 (Memory based)

30 January 2023 - Shift 1

Answer & Solutions

MATHEMATICS

1. Coefficient of x^{301} in $(1+x)^{500}+x(1+x)^{499}+x^2(1+x)^{498}+\cdots+x^{500}$ is equal to

A. $^{506}C_{306}$
B. $^{501}C_{300}$
C. $^{501}C_{301}$
D. $^{500}C_{300}$

Answer (C)
Solution:

Coefficient of $x^{301}={}^{500}C_{301}+{}^{499}C_{300}+{}^{498}C_{299}+\cdots+{}^{199}C_0$

$={}^{500}C_{199}+{}^{499}C_{199}+{}^{498}C_{199}+\cdots+{}^{199}C_{199}$

$={}^{501}C_{200}$

$={}^{501}C_{301}$ $\cdots$ (since $^nC_r={}^nC_{n-r}$)

2. $\tan 15°+\frac{1}{\tan 165°}+\frac{1}{\tan 105°}+\tan 195°=2a$. Then the value of $\left(a+\frac{1}{a}\right)$ is __________.

A. $4-2\sqrt{3}$

B. $-\frac{4}{\sqrt{3}}$

C. 2

D. $5-\frac{3}{2}\sqrt{3}$

Answer (B)

Solution:

$\tan 15°+\cot 165°+\cot 105°+\tan 195°$

$=\tan 15°-\cot 15°-\tan 15°+\tan 15°$

$=\tan 15°-\cot 15°$

$=-2\sqrt{3}$

$\Rightarrow a=-\sqrt{3}$ $\Rightarrow a+\frac{1}{a}=-\sqrt{3}-\frac{1}{\sqrt{3}}=-\frac{4}{\sqrt{3}}$

3. If set $A=\{a,\ b,\ c\}$, $R: A\to A$, $R=\{(a,\ b),\ (b,\ c)\}$. How many elements should be added for making it symmetric and transitive?

A. 2
B. 3
C. 4
D. 7

Answer (D)
Solution:

For symmetric

$(a,\ b),\ (b,\ c)\in R$

$\Rightarrow (b, a), (c, b) \in R$

For Transitive

$(a, b), (b, c) \in R$

$\Rightarrow (a, c) \in R$

Now,

$(a, c) \in R$

$(c, a) \in R$ (for symmetric)

$(a, b), (b, a) \in R$

$\Rightarrow (a, a) \in R$

$(b, c), (c, b) \in R$

$\Rightarrow (b, b) \in R$

$(c, b), (b, c) \in R$

$\Rightarrow (c, c) \in R$

$\therefore$ elements to be added

$\{(b, a), (c, b), (b, b), (a, a), (a, c), (c, a), (c, c)\}$

Total 7 elements

4. Let $P(h, k)$ be any point on $x^2 = 4y$ which is at shortest distance from $Q(0, 33)$, then difference of distances of $P(h, k)$ from directrix of $y^2 = 4(x + y)$ is:

A. 2

B. 4

C. 6

D. 8

Answer (B)

Solution:

For normal through $Q(0, 33)$

Normal at point $(2t, t^2)$

$x = -ty + 2at + at^3$

$\Rightarrow 0 = -t \cdot 33 + 2a + t^3$

$\Rightarrow t = 0$ or $\pm\sqrt{31}$

Points at which normal are drawn are

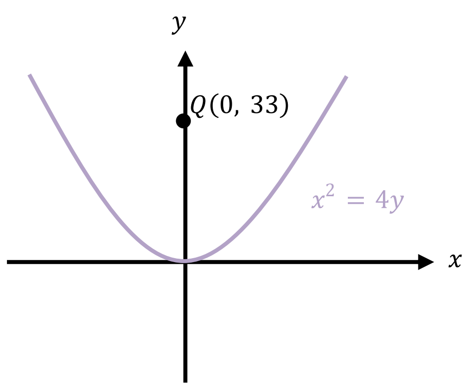
$A(0, 0), B(2\sqrt{31}, 31), C(-2\sqrt{31}, 31)$

S.D $= QB = QC = \sqrt{124 + 4} = 8\sqrt{2}$ units

Given parabola $(y - 2)^2 = 4(x + 1)$

Directrix is $x = -2$, that is line L

$B_l - C_l = \left|\left(-2 + 2\sqrt{31}\right) - \left(2 + 2\sqrt{31}\right)\right| = 4$



5. Area bounded by larger part in I^{st} quadrant by $x = 4y^2$, $x = 2$ and $y = x$ is A, then $3A$ equals :

A. $6 + \frac{1}{32} - 2\sqrt{2}$

B. $2 + \frac{1}{96} - \frac{2\sqrt{2}}{3}$

C. $\frac{2\sqrt{2}}{3}$

D. 96

Answer (A)

Solution:

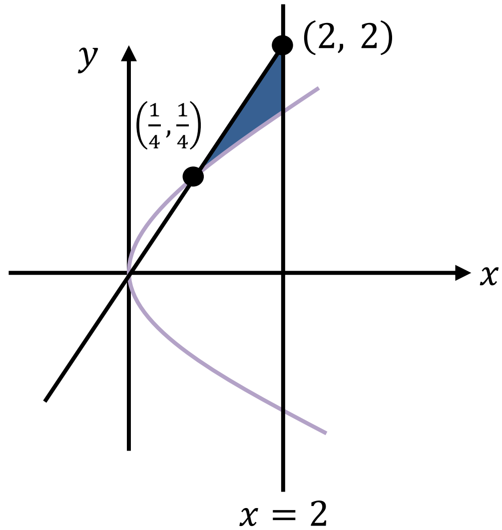
$A = \int_{\frac{1}{4}}^{2} \left(x - \frac{\sqrt{x}}{2}\right) dx$

$= \frac{x^2}{2} - \frac{x^{\frac{3}{2}}}{3}\Big|_{\frac{1}{4}}^{2}$

$= \left(2 - \frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{32} - \frac{1}{24}\right)$

$= 2 + \frac{1}{96} - \frac{2\sqrt{2}}{3}$

$\Rightarrow 3A = 6 + \frac{1}{32} - 2\sqrt{2}$ sq. units



6. A die with points $(2, 1, 0, -1, -2, 3)$ is thrown 5 times. The probability that the product of outcomes on all throws is positive is _______.

Answer ($\frac{521}{2592}$)

Solution:

$(2, 1, 0, -1, -2, 3)$

P(positive number) $= \frac{3}{6} = \frac{1}{2}$

P(negative number) $= \frac{2}{6} = \frac{1}{3}$

$E =$ product is positive

$E =$ (5 positive (or) 3 positive 2 negative (or) 1 positive 4 negative)

$P(E) = {}^5C_5 \left(\frac{1}{2}\right)^5 + {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 + {}^5C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{3}\right)^4$

$= \frac{1}{32} + \frac{5}{36} + \frac{5}{81 \cdot 2} = \frac{521}{2592}$

7. Let $S\{1, 2, 3, 4, 5\}$. If $f: S \to P(S)$, where $P(S)$ is power set of S. Then number of one-one function f can be made is:

A. $(32)^5$

B. $\frac{32!}{27!}$

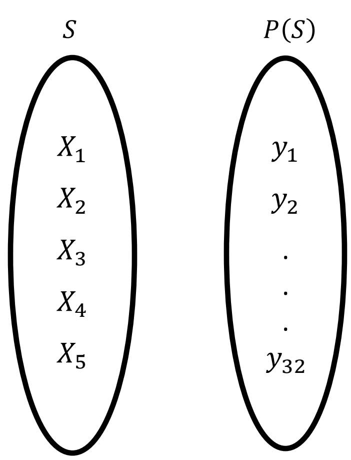
C. ${}^{32}C_{27}$

D. ${}^{32}P_{27}$

Answer (B)
Solution:

$n(S) = 5$

$n(P(S)) = 2^5 = 32$



∴ No. of one-one function

$= 32 \times 31 \times 30 \times 29 \times 28$

$= \frac{32!}{27!}$

8. A line is cutting x-axis and y-axis at two points A and B respectively, where $OA = a,\ OB = b$. A perpendicular is drawn is drawn from O (origin) to AB at an angle of $\frac{\pi}{6}$ from positive x-axis. If area of triangle $OAB = \frac{98\sqrt{3}}{3}$ sq. units, then $\sqrt{3}a + b$ is equal to:

A. 28

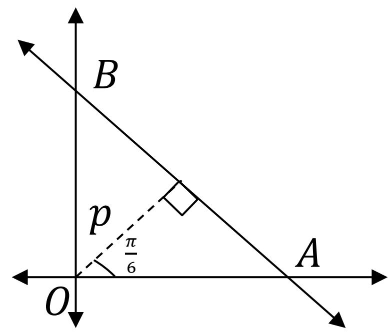
B. 14

C. 12

D. 7

Answer (A)

Solution:



Let the perpendicular distance of line from origin is p.

⇒ Equation of AB: $x \cos\frac{\pi}{6} + y \sin\frac{\pi}{6} = p$

$\Rightarrow \frac{x\sqrt{3}}{2} + \frac{y}{2} = p$

$\Rightarrow \frac{x}{\frac{2p}{\sqrt{3}}} + \frac{y}{2p} = 1$

Compare above equation to intercept form of line $\frac{x}{a} + \frac{y}{b} = 1$

$OA = a = \frac{2p}{\sqrt{3}},\ OB = b = 2p$

∴ Area of triangle $OAB = \frac{1}{2} \cdot \frac{2p}{\sqrt{3}} \cdot 2p = \frac{98\sqrt{3}}{3}$

$\Rightarrow p = 7$

$OA = a = \frac{14}{\sqrt{3}}$

$OB = b = 14$

$\Rightarrow \sqrt{3}a + b = 14 + 14 = 28$

9. $\frac{3(e-1)}{e}\int_1^2 x^2 e^{[x]+[x^3]}dx$ equals:

A. $e^9 - e$

B. $e^8 - 1$

C. $e^8 - e$

D. $e^9 - 1$

Answer (C)

Solution:

$I = \int_1^2 x^2 e^{[x]+[x^3]}dx = e\int_1^2 x^2 e^{[x^3]}dx$

Let $x^3 = t$

$I = e\int_1^8 \frac{dt}{3} \cdot e^{[t]} = \frac{e}{3}(e + e^2 + \cdots e^7)$

$= \frac{e^2}{3}\left(\frac{e^7-1}{e-1}\right)$

So $\frac{3(e-1)}{e} \cdot \frac{e^2}{3} \cdot \frac{e^7-1}{e-1} = e^8 - e$

10. $\vec{n}$ is a vector, $\vec{a} \neq 0$, $\vec{b} \neq 0$. If $\vec{n} \perp \vec{c}$, $\vec{a} = \alpha\vec{b} - \hat{n}$ and $\vec{b} \cdot \vec{c} = 12$, then the value of $\left|\vec{c} \times \left(\vec{a} \times \vec{b}\right)\right|$ equals:
(where $\hat{n}$ represents unit vector in direction of $\vec{n}$)

A. 144
B. $\sqrt{12}$
C. 12
D. 24

Answer (C)

Solution:

$\vec{a} = \alpha\vec{b} - \hat{n}$
$\Rightarrow \vec{a} \times \vec{b} = -\hat{n} \times \vec{b}$
$\left|\vec{c} \times \left(\vec{a} \times \vec{b}\right)\right| = \left|\vec{c} \times \left(-\hat{n} \times \vec{b}\right)\right| = \left|-\hat{n}\left(\vec{c} \cdot \vec{b}\right) - \vec{b}\left(\vec{c} \cdot (-\hat{n})\right)\right|$
$\Rightarrow \left|\vec{c} \times \left(\vec{a} \times \vec{b}\right)\right| = \left|-\hat{n}(12) - \vec{b}(0)\right| = 12$

11. $\lim_{x \to 0} \frac{48 \int_0^x \frac{t^3}{1+t^6} dt}{x^4}$ equals ___________.

Answer (12)

Solution:

$\lim_{x \to 0} \frac{48 \int_0^x \frac{t^3}{1+t^6} dt}{x^4}$
As $\frac{0}{0}$ form, applying L'hospital rule we get,
$\lim_{x \to 0} \frac{48\, x^3}{(x^6+1) \cdot 4x^3} = 48 \cdot \frac{1}{4} = 12$

12. If $a_n = \frac{-2}{4n^2 - 16n + 15}$, and $a_1 + a_2 + \cdots + a_{25} = \frac{m}{n}$ where m and n are coprime, then the value of $m + n$ is ______.

Answer (191)

Solution:
We have,
$a_n = \frac{-2}{4n^2 - 16n + 15}$
$= \frac{-2}{(2n-3)(2n-5)}$
$= \frac{1}{(2n-3)} - \frac{1}{(2n-5)}$
$\Rightarrow a_1 + a_2 + \cdots + a_{25} = \left(\frac{1}{-1} - \frac{1}{-3}\right) + \cdots + \left(\frac{1}{47} - \frac{1}{45}\right)$
$= \frac{1}{47} + \frac{1}{3}$
$= \frac{50}{141} = \frac{m}{n}$
$\Rightarrow m + n = 191$

13. If $z = 1 + i$ and $z_1 = \frac{i + \bar{z}(1-i)}{\bar{z}(1-z)}$, then the value of $\frac{12}{\pi} \arg(z_1)$ is ______.

Answer (3)

Solution:
$z_1 = \frac{i + \bar{z}(1-i)}{\bar{z}(1-z)}$
$= \frac{i + (1-i)(1-i)}{(1-i)(-i)}$

$= \frac{i-2i}{(1-i)(-i)}$
$= \frac{1}{1-i}$
$\arg(z_1) = \arg\left(\frac{1}{1-i}\right) = \arg\left(\frac{1+i}{(1-i)(1+i)}\right) = \arg\left(\frac{1}{2} + \frac{i}{2}\right) = \frac{\pi}{4}$
$\frac{12}{\pi}\arg(z_1) = \frac{12}{\pi} \times \frac{\pi}{4} = 3$

14. Mean and variance of 7 observations are 8 & 16 respectively. If number 14 is omitted, then a & b are new mean and variance. The value of $a + b$ is ______.

Answer (19)

Solution:

Let $x_1, x_2, \ldots, x_7$ be the 7 observations
New mean $(a) = \frac{8\times 7-14}{6} = 7$
$\frac{\sum_{i=1}^{7} x_i^2}{7} - 64 = 16$
$\Rightarrow \sum x_i^2 = 560$
$\sum x_{i\ new}^2 = 560 - 14^2$
$\therefore b = \frac{364}{6} - 7^2 = \frac{70}{6} = \frac{35}{3}$
$a = 7; b = \frac{35}{3}$
$a + b = 7 + \frac{35}{3} = \frac{56}{3} = 18.67 \approx 19$ (Rounding off gives 19)

15. If coefficient of x^{15} in expansion of $\left(ax^3 + \frac{1}{bx^{\frac{1}{3}}}\right)^{15}$ is equal to coefficient of x^{-15} in expansion of $\left(ax^{\frac{1}{3}} + \frac{1}{bx^3}\right)^{15}$, then $|ab - 5|$ is equal to _________.

Answer (4)

Solution:

For expansion of $\left(ax^3 + \frac{1}{bx^{\frac{1}{3}}}\right)^{15}$
$T_{r+1} = {}^{15}C_r \times a^{15-r} \times (x^3)^{15-r} \times b^{-r} \times x^{\frac{-r}{3}}$
Here we need coefficient of x^{15}
$\Rightarrow 45 - 3r - \frac{r}{3} = 15$
$\Rightarrow \frac{10r}{3} = 30$
$\Rightarrow r = 9$
$\therefore$ Coefficient of x^{15} in $\left(ax^3 + \frac{1}{bx^{\frac{1}{3}}}\right)^{15} = {}^{15}C_9 \times a^6 \times b^{-9}$

For expansion of $\left(ax^{\frac{1}{3}} + \frac{1}{bx^3}\right)^{15}$
$T_{r+1} = {}^{15}C_r \times a^{15-r} \times (x)^{\frac{15-r}{3}} \times b^{-r} \times x^{-3r}$
Here we need coefficient of x^{-15}
$\Rightarrow \frac{15-r}{3} - 3r = -15$
$\Rightarrow 15 - r - 9r = -45$
$\Rightarrow r = 6$
$\therefore$ Coefficient of x^{-15} in $\left(ax^{\frac{1}{3}} + \frac{1}{bx^3}\right)^{15} = {}^{15}C_6 \times a^9 \times b^{-6}$
Coefficient of x^{15} in $\left(ax^3 + \frac{1}{bx^{\frac{1}{3}}}\right)^{15}$ = Coefficient of x^{-15} in $\left(ax^{\frac{1}{3}} + \frac{1}{bx^3}\right)^{15}$ $\cdots$ (given)
$\Rightarrow a^{-3}b^{-3} = 1$
$\Rightarrow ab = 1$
$\Rightarrow |ab - 5| = |1 - 5| = 4$

16. Using $1,2,3$ and 5, four digit numbers are formed, where repetition is allowed. The number of numbers divisible by 15 are _______.

Answer (21)

Solution:

Unit digit will be 5
$\underline{a}\ \ \underline{b}\ \ \underline{c}\ \ \underline{5}$
$a+b+c=(3\lambda+1)$ type
For (a,b,c) possibilities are
$(2,2,3)\ (1,1,5)\ (1,1,2)\ (3,3,1)\ (5,5,3)\ (2,3,5)$
For $(2,2,3)\ \Rightarrow \frac{3!}{2!}=3$
For $(1,1,5)\Rightarrow \frac{3!}{2!}=3$
For $(1,1,2)\Rightarrow \frac{3!}{2!}=3$
For $(3,3,1)\Rightarrow \frac{3!}{2!}=3$
For $(5,5,3)\Rightarrow \frac{3!}{2!}=3$
For $(2,3,5)\Rightarrow \frac{3!}{2!}=3$
Total $=21$

17. If $5f(x+y)=f(x)\cdot f(y)$ and $f(3)=320$, then the value of $f(1)$ is ______.

Answer (20)

Solution:

$5f(x+y)=f(x)\cdot f(y)\ \ \cdots(1)$
Put $x=1,\ y=2$ in (1)
$5f(3)=f(1)\cdot f(2)$
$\Rightarrow f(1)\cdot f(2)=5\times 320=1600\ \ \cdots(2)$
Put $x=y=1$ in (1)
$f(2)=\frac{(f(1))^2}{5}\ \ \cdots(3)$
$f(3)=320$
Using (2) and (3),
$f(1)\cdot\frac{(f(1))^2}{5}=1600$
$(f(1))^3=8000$
$f(1)=20$

18. If for $\log_{\cos x}(\cot x)-4\log_{\sin x}(\cot x)=1,\ x=\sin^{-1}\left(\frac{\alpha+\sqrt{\beta}}{2}\right)$. Then the value of $(\alpha+\beta)$ is __________.
(given $x\in\left(0,\frac{\pi}{2}\right)$)

Answer (4)

Solution:

$\log_{\cos x}(\cot x)-4\log_{\sin x}(\cot x)=1$
$\Rightarrow 1-\log_{\cos x}(\sin x)-4(\log_{\sin x}(\cos x)-1)=1$
Let $\log_{\cos x}(\sin x)=t$
$\Rightarrow -t-4\left(\frac{1}{t}-1\right)=0$
$\Rightarrow t+\frac{4}{t}=4$
$\Rightarrow t=2$
$\therefore t=\log_{\cos x}(\sin x)=2$
$\Rightarrow \cos^2 x=sin\,x$
$\Rightarrow 1-\sin^2 x=\sin x$

$\Rightarrow \sin^2 x + \sin x - 1 = 0$
$\Rightarrow \sin x = \frac{-1 \pm \sqrt{5}}{2}$
Comparing, $\alpha = -1,\ \beta = 5$
$\Rightarrow \alpha + \beta = 4$